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A COMPARISON OF AUTOMORPHIC L-FUNCTIONS IN A THETA SERIES LIFTING FOR UNITARY GROUPS

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ABSTRACT

A relation between automorphic L-functions for $U(n,n+1) \times \mathrm{GL}(n)$ and those for $U(n,n) \times \mathrm{GL}(n)$ in a theta series lifting is studied by using the Rankin–Selberg method.

Introduction

Let E be a quadratic extension of an algebraic number field F and let $G = U(n,n)$ (resp. $G^* = U(n,n+1)$) be a quasi-split unitary group with respect to a skew Hermitian form on E^{2n} (resp. a Hermitian form on E^{2n+1}) with Witt index n. We denote by $\mathrm{R}_{E/F}(\mathrm{GL}(n))$ the scalar restriction of $\mathrm{GL}(n)$ defined over E. For irreducible cuspidal automorphic representations σ, σ^* and π of $G(\mathbb{A})$, $G^*(\mathbb{A})$ and $\mathrm{R}_{E/F}(\mathrm{GL}(n))(\mathbb{A})$, respectively, one can define the (partial) automorphic L-functions $L_S(s, \sigma \times \pi)$ and $L_S(s, \sigma^* \times \pi)$ by Euler products on some right half plane $\mathrm{Re}(s) >> 0$. If σ and σ^* are generic, it is known by Shahidi [Sh] that these L-functions are analytically continued to meromorphic functions on the whole s-plane. On the other hand, there is a class of automorphic representations of $G^*(\mathbb{A})$ lifted from the cuspidal automorphic representations of $G(\mathbb{A})$. This lifting depends on the choice of a nontrivial character μ of $F\backslash\mathbb{A}$ and a certain Hecke character ν of $E^\times\backslash\mathbb{A}_E^\times$. We denote by $\Theta^n_{\mu,\nu}(\sigma)$ a lifting of σ. It is known by [Wa1] that if σ is generic, then $\Theta^n_{\mu,\nu}(\sigma)$ is nonzero and generic for some choice of μ. The cuspidality condition of $\Theta^n_{\mu,\nu}(\sigma)$ was also given in [Wa1]. Then a natural question is: How does $L_S(s, \Theta^n_{\mu,\nu}(\sigma) \times \pi)$ relate with $L_S(s, \sigma \times \pi)$? Our main theorem answers this question.

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THEOREM: *Assume that $\Theta^n_{\mu,\nu}(\sigma)$ is cuspidal and irreducible. Then one has*

$$L_S(s, \Theta^n_{\mu,\nu}(\sigma) \times \pi) = L_S(s, \overline{\pi}) L_S(s, \sigma \times (\overline{\pi} \otimes \nu)),$$

where $\overline{\pi}$ is the twist of π by the Galois involution of E/F, i.e. $\overline{\pi}(g) = \pi(\overline{g})$, and $L_S(s, \overline{\pi})$ is the standard L-function of $\overline{\pi}$ regarded as a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$.

We expect that such a relation between L-functions has an application to the characterization of the lifting image. (cf. [G-P2])

To prove this theorem, we need integral representations of L-functions. The Rankin–Selberg method for the group of type $G \times \mathrm{GL}(n)$ was established by Gelbart and Piatetski-Shapiro [G-P] when G is a classical split group of rank n. Their method can also be applied to the cases $G = U(n,n)$ and $G = U(n, n+1)$. The case $G = U(n, n+1)$ essentially treated by Tamir ([Ta]) and the case $G = U(n,n)$ is similarly investigated as the case of $G = \mathrm{Sp}(n)$. However, since we could not find adequate references for calculations of ramified primes of local integrals, we included necessary calculations of archimedean and finite ramified local integrals in this paper.

We organize this paper as follows. In Sections 1 and 2, we prepare some notations and define the L-functions $L_S(s, \sigma \times \pi)$ and $L_S(s, \sigma^* \times \pi)$. In Section 3, we recall Eisenstein series, Theta series and basic identities deduced from Theorems A and C in [G-P]. In Section 4, the related local integrals are calculated and integral representations of L-functions completed. The main theorem is proved in Section 5.

1. Notations

For an associative ring R with identity element, we denote by $R^\times$ the group of all invertible elements of R and by $M_{n,m}(R)$ the set of all $n \times m$ matrices with entries in R. If $n = m$, we write $M_n(R)$ for $M_{n,n}(R)$. For $A \in M_{n,m}(R)$, ${}^tA$ stands for its transpose. For $A \in M_n(R)$, $\det A$ and $\mathrm{Tr}\, A$ stand for its determinant and trace, respectively. The identity matrix in $M_n(R)$ is denoted by 1_n.

Let F be an algebraic number field and $\mathbf{V}_F$ the set of all places of F. For $v \in \mathbf{V}_F$, F_v stands for the completion of F at v. If v is a finite place, $\mathcal{O}_v$ denotes the ring of integers in F_v, $\mathcal{P}_v$ the maximal ideal of $\mathcal{O}_v$ and q_v the order of the residual field $\mathcal{O}_v/\mathcal{P}_v$. The ring of adeles of F is denoted by $\mathbb{A}$. Let E be a quadratic extenstion of F. The Galois involution of E over F is denoted by a bar or ϵ. We write $\mathbb{A}_E$ for the ring of adeles of E. The norm and the trace of E

over F are denoted by $\mathrm{N}_{E/F}$ and $\mathrm{Tr}_{E/F}$, respectively. Throughout this paper, we fix a nontrivial additive character μ of $F\backslash\mathbb{A}$ and a character ν of $E^\times\backslash\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}^\times$ equals the quadratic character associated to E/F by class field theory.

If H is an F-algebraic group and R an F-algebra, $H(R)$ denotes the group of R-rational points of H. We often consider H as an algebraic group defined over E and denote by $\mathrm{R}_{E/F}(H)$ the scalar restriction of H, hence one has $\mathrm{R}_{E/F}(H)(R) = H(R\otimes E)$. The Galois group of E over F naturally acts on $\mathrm{R}_{E/F}(H)$.

We will use the following notations for F-subgroups of GL_n.

B_0 the Borel subgroup of GL_n consisting of upper triangular matrices,

T_0 the maximal torus consisting of diagonal matrices in GL_n,

Δ_0 the unipotent radical of B_0,

Q_0 the stabilizer in GL_n of the vector $(0,\dots,0,1)\in M_{1,n}(F)$,

Z_0 the center of GL_n.

We define F-algebraic groups G and G^* by

$$G = \left\{g\in \mathrm{R}_{E/F}(\mathrm{GL}_{2n})\colon {}^t g J_n \bar{g} = J_n\right\}, \qquad J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0\end{pmatrix},$$

$$G^* = \left\{g\in \mathrm{R}_{E/F}(\mathrm{GL}_{2n+1})\colon {}^t g J_n^* \bar{g} = J_n^*\right\}, \quad J_n^* = \begin{pmatrix} 0 & & 1_n \\ & 1 & \\ 1_n & & 0\end{pmatrix}.$$

In this paper, we fix an element $\mathbf{i}\in E^\times$ with $\mathbf{i}+\bar{\mathbf{i}}=0$, and an embedding

$$\iota\colon G\hookrightarrow G^*\colon \begin{pmatrix} A & B \\ C & D\end{pmatrix} \mapsto \begin{pmatrix} A & 0 & \mathbf{i}B \\ 0 & 1 & 0 \\ \mathbf{i}^{-1}C & 0 & D\end{pmatrix}.$$

We will consider the following F-subgroups of G^*:

$$M^* = \left\{ m^*(a,\varepsilon) = \begin{pmatrix} a & & \\ & \varepsilon & \\ & & {}^t\bar{a}^{-1}\end{pmatrix} \ :\ \begin{matrix} a\in \mathrm{R}_{E/F}(\mathrm{GL}_n) \\ \varepsilon\in \mathrm{R}_{E/F}(\mathrm{GL}_1),\ \varepsilon\bar{\varepsilon}=1\end{matrix}\right\},$$

$$T^* = \left\{m^*(t,\varepsilon)\colon t\in \mathrm{R}_{E/F}(T_0),\ \varepsilon\in \mathrm{R}_{E/F}(\mathrm{GL}_1),\ \varepsilon\bar{\varepsilon}=1\right\},$$

$$\Delta^* = \left\{m^*(\delta,1)\colon \delta\in \mathrm{R}_{E/F}(\Delta_0)\right\},$$

$$N^* = \left\{ v^*(b,c) = \begin{pmatrix} 1_n & c & b - c\,{}^t\bar{c}/2 \\ & 1 & -{}^t\bar{c} \\ & & 1_n\end{pmatrix} \ :\ \begin{matrix} b\in \mathrm{R}_{E/F}(M_n),\ {}^t\bar{b} = -b \\ c\in \mathrm{R}_{E/F}(M_{n,1})\end{matrix}\right\}.$$

Then $P^* = M^* \ltimes N^*$ is the standard maximal parabolic subgroup of G^* which preserves a maximal isotropic subspace, $U^* = \Delta^* \ltimes N^*$ is a maximal unipotent subgroup of G^* and $B^* = T^* \ltimes U^*$ is a Borel subgroup of G^*. If H^* is one of the groups defined above, its inverse image $\iota^{-1}(H^*)$ is denoted by H, so that $P = M \ltimes N$ is a Siegel type parabolic subgroup and $B = T \ltimes U$ a Borel subgroup of G. The Levi subgroup M is identified with $\mathrm{R}_{E/F}(\mathrm{GL}_n)$. We will use the following notations for elements of $M(\mathbb{A})$ and $N(\mathbb{A})$:

$$m(a) = \begin{pmatrix} a & \\ & {}^t\overline{a}^{-1} \end{pmatrix} \quad (a \in \mathrm{R}_{E/F}(\mathrm{GL}_n)(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_E)),$$
$$v(b) = \begin{pmatrix} 1_n & b \\ & 1_n \end{pmatrix} \quad (b \in \mathrm{R}_{E/F}(M_n)(\mathbb{A}) = M_n(\mathbb{A}_E), \ \ {}^t\overline{b} = b).$$

The standard maximal compact subgroups of $\mathrm{GL}_n(\mathbb{A})$, $R_{E/F}(\mathrm{GL}_n)(\mathbb{A})$, $G(\mathbb{A})$ and $G^*(\mathbb{A})$ will be denoted by $K_0 = \prod_v K_{0,v}$, $K_1 = \prod_v K_{1,v}$, $K = \prod_v K_v$ and $K^* = \prod_v K_v^*$, respectively. We define nondegenerate characters ψ_0, ψ and ψ^* of $\Delta_0(\mathbb{A})$, $U(\mathbb{A})$ and $U^*(\mathbb{A})$, respectively, as follows:

$$\psi_0(\delta) = \mu(\delta_{1\,2} + \delta_{2\,3} + \cdots + \delta_{n-1\,n}), \qquad (\delta = (\delta_{ij}) \in \Delta_0(\mathbb{A})),$$
$$\psi(u) = \mu(\mathrm{Tr}_{E/F}(u_{1\,2} + u_{2\,3} + \cdots + u_{n-1\,n}) - u_{n\,2n}), \quad (u = (u_{ij}) \in U(\mathbb{A})),$$
$$\psi^*(u^*) = \mu(\mathrm{Tr}_{E/F}(u^*_{1\,2} + u^*_{2\,3} + \cdots + u^*_{n\,n+1})), \qquad (u^* = (u^*_{ij}) \in U^*(\mathbb{A})).$$

The restriction of ψ to $\Delta(\mathbb{A})$ is denoted by ψ_Δ.

Throughout this paper, we fix irreducible cuspidal automorphic representations (σ, V_σ), (σ^*, V_{σ^*}) and (π, V_π) of $G(\mathbb{A})$, $G^*(\mathbb{A})$ and $M(\mathbb{A}) \cong \mathrm{GL}_n(\mathbb{A}_E)$, respectively. For cusp forms $\varphi \in V_\sigma$, $\varphi^* \in V_{\sigma^*}$ and $\Phi \in V_\pi$, we define

$$W_\varphi(g) = \int_{U(F)\backslash U(\mathbb{A})} \psi(u)^{-1}\varphi(ug)du,$$
$$W_{\varphi^*}(g^*) = \int_{U^*(F)\backslash U^*(\mathbb{A})} \psi^*(u^*)^{-1}\varphi^*(u^*g^*)du^*,$$
$$W_\Phi(m) = \int_{\Delta(F)\backslash \Delta(\mathbb{A})} \psi_\Delta(\delta)\Phi(\delta m)d\delta.$$

Then Whittaker models of σ, σ^* and π are given by

$$W(\sigma, \psi) = \{W_\varphi \colon \varphi \in V_\sigma\}, \quad W(\sigma^*, \psi^*) = \{W_{\varphi^*} \ : \ \varphi^* \in V_{\sigma^*}\},$$
$$W(\pi, \psi_\Delta^{-1}) = \{W_\Phi \colon \Phi \in V_\pi\}.$$

We assume that both σ and σ^* are generic, i.e. $W(\sigma,\psi)\neq 0$ and $W(\sigma^*,\psi^*)\neq 0$. Let $\omega_\sigma,\omega_{\sigma^*}$ and ω_π be central characters of σ,σ^* and π, respectively. There is a unique real number r_π such that $|\omega_\pi(m(a1_n))| = |a|_{\mathbb{A}_E}^{-r_\pi/n}$ for any $a\in\mathbb{A}_E^\times$. Then $\pi\otimes|\det(\cdot)|_{\mathbb{A}_E}^{r_\pi}$ becomes a unitary cuspidal automorphic representation.

For each $v\in\mathbf{V}_F$, we denote the corresponding local Whittaker models by $W(\sigma_v,\psi_v)$, $W(\sigma_v^*,\psi_v^*)$ and $W(\pi_v,\psi_{\Delta,v}^{-1})$. If v is archimedean, we will often use the conventions of [J-S3]. Let $\mathfrak{g}_v$ be the Lie algebra of $G(F_v)$. By a theorem of Casselman and Wallach [C], the irreducible $(\mathfrak{g}_v,K_v)$-module V_{σ_v} can be realized as the $(\mathfrak{g}_v,K_v)$-module coming from a continuous representation of $G(F_v)$ on a Frechet space $V_{\sigma_v}^\infty$, which is smooth and with moderate growth. This continuous representation will be also denoted by σ_v. There is a unique (up to constant) continuous Whittaker functional λ_v on $V_{\sigma_v}^\infty$. We denote by $W^\infty(\sigma_v,\psi_v)$ the space of all functions on $G(F_v)$ of the form $g\mapsto\lambda_v(\sigma_v(g)x)$, $x\in V_{\sigma_v}^\infty$. Then $W(\sigma_v,\psi_v)$ is a subspace consisting of all K_v-finite vectors in $W^\infty(\sigma_v,\psi_v)$. To unify the notations, for a finite $v\in\mathbf{V}_F$, we sometime write $V_{\sigma_v}^\infty$ and $W^\infty(\sigma_v,\psi_v)$ for V_{σ_v} and $W(\sigma_v,\psi_v)$, respectively. In a similar fashion, we define $W^\infty(\sigma_v^*,\psi_v^*)$ and $W^\infty(\pi_v,\psi_{\Delta,v}^{-1})$ for each $v\in\mathbf{V}_F$.

We will denote by $\mathbf{V}_F(\sigma,\sigma^*,\pi,\mu,\nu)$ the set of finite places $v\in\mathbf{V}_F$ such that all data σ_v, σ_v^*, π_v, μ_v and ν_v are unramified. The complement of $\mathbf{V}_F(\sigma,\sigma^*,\pi,\mu,\nu)$ in $\mathbf{V}_F$ will be denoted by S.

2. Definition of L-functions

We recall a definition of partial automorphic L-functions $L_S(s,\sigma\times\pi)$ and $L_S(s,\sigma^*\times\pi)$. Let $\Gamma_{E/F}=\{e,\epsilon\}$ be the Galois group of E over F. L-groups of $G\times M$ and $G^*\times M$ are

$$
\begin{aligned}
{}^L(G\times M) &= (\mathrm{GL}_{2n}(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C}))\rtimes\Gamma_{E/F},\\
{}^L(G^*\times M) &= (\mathrm{GL}_{2n+1}(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C}))\rtimes\Gamma_{E/F},
\end{aligned}
$$

respectively. Let ρ_m (resp. ρ_m^0) be the standard (resp. trivial) representation of $\mathrm{GL}_m(\mathbb{C})$. Then $\rho_{2n}\otimes\rho_n\otimes\rho_n^0$ (resp. $\rho_{2n+1}\otimes\rho_n\otimes\rho_n^0$) is a representation of the group ${}^L(G\times M)^0$ (resp. ${}^L(G^*\times M)^0$) of the identity component of ${}^L(G\times M)$ (resp. ${}^L(G^*\times M)$). We denote by r (resp. r^*) the representation of ${}^L(G\times M)$ (resp. ${}^L(G^*\times M)$) induced from $\rho_{2n}\otimes\rho_n\otimes\rho_n^0$ (resp. $\rho_{2n+1}\otimes\rho_n\otimes\rho_n^0$).

If $v\in\mathbf{V}_F\backslash S$ remains prime in E and Satake parameters of σ_v, σ_v^* and π_v are given by $(\alpha_{v,1},\dots,\alpha_{v,n})$, $(\alpha_{v,1}^*,\dots,\alpha_{v,n}^*)$ and $(\beta_{v,1},\dots,\beta_{v,n})$, respectively, then we put

$$\gamma_v = \left(\begin{pmatrix} \alpha_{v,1} & & & \\ & \ddots & & \\ & & \alpha_{v,n} & \\ & & & 1_n \end{pmatrix}, \begin{pmatrix} \beta_{v,1} & & \\ & \ddots & \\ & & \beta_{v,n} \end{pmatrix}, 1_n \right) \rtimes \epsilon,$$

$$\gamma_v^* = \left(\begin{pmatrix} \alpha^*_{v,1} & & & \\ & \ddots & & \\ & & \alpha^*_{v,n} & \\ & & & 1_{n+1} \end{pmatrix}, \begin{pmatrix} \beta_{v,1} & & \\ & \ddots & \\ & & \beta_{v,n} \end{pmatrix}, 1_n \right) \rtimes \epsilon.$$

If $v \in \mathbf{V}_F \backslash S$ splits in E and Satake parameters of σ_v, σ_v^* and π_v are given by $(\alpha_{v,1}, \cdots, \alpha_{v,2n})$, $(\alpha^*_{v,1}, \cdots, \alpha^*_{v,2n+1})$ and $((\beta_{v,1}, \cdots, \beta_{v,n}), (\beta'_{v,1}, \cdots, \beta'_{v,n}))$, respectively, then we put

$$\gamma_v = \left(\begin{pmatrix} \alpha_{v,1} & & \\ & \ddots & \\ & & \alpha_{v,2n} \end{pmatrix}, \begin{pmatrix} \beta_{v,1} & & \\ & \ddots & \\ & & \beta_{v,n} \end{pmatrix}, \begin{pmatrix} \beta'_{v,1} & & \\ & \ddots & \\ & & \beta'_{v,n} \end{pmatrix} \right) \rtimes e\,,$$

$$\gamma_v^* = \left(\begin{pmatrix} \alpha^*_{v,1} & & \\ & \ddots & \\ & & \alpha^*_{v,2n+1} \end{pmatrix}, \begin{pmatrix} \beta_{v,1} & & \\ & \ddots & \\ & & \beta_{v,n} \end{pmatrix}, \begin{pmatrix} \beta'_{v,1} & & \\ & \ddots & \\ & & \beta'_{v,n} \end{pmatrix} \right) \rtimes e.$$

The Euler factors $L_v(s, \sigma_v \times \pi_v)$ and $L_v(s, \sigma_v^* \times \pi_v)$ are defined to be

$$L_v(s, \sigma_v \times \pi_v) = \det(1_{4n^2} - r(\gamma_v) q_v^{-s})^{-1},$$
$$L_v(s, \sigma^*_v \times \pi_v) = \det(1_{2n(2n+1)} - r^*(\gamma^*_v) q_v^{-s})^{-1}.$$

It is known that both Euler products

$$L_S(s, \sigma \times \pi) = \prod_{v \notin S} L_v(s, \sigma_v \times \pi_v), \qquad L_S(s, \sigma^* \times \pi) = \prod_{v \notin S} L_v(s, \sigma^*_v \times \pi_v)$$

are absolutely convergent in $\mathrm{Re}(s) >> 0$. In the following sections, we will prove that both L-functions $L_S(s, \sigma \times \pi)$ and $L_S(s, \sigma^* \times \pi)$ have analytic continuations to meromorphic functions on the whole s-plane.

3. Basic identities

In this section, we define Eisenstein series and theta series on $G(\mathbb{A})$, and recall the basic identities. Let $\alpha\colon M(\mathbb{A}) \to \mathbb{C}^\times$ be the quasicharacter defined as $\alpha(m(a)) =$

$|\det a|_{\mathbb{A}_E}$. By using the Iwasawa decomposition, the quasicharacter α is extended to a function on $G(\mathbb{A})$, i.e. $\alpha(umk) = \alpha(m)$ for $u \in N(\mathbb{A})$, $m \in M(\mathbb{A})$ and $k \in K$. For each $s \in \mathbb{C}$, we denote by $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi \otimes \alpha^s$ the representation space of $G(\mathbb{A})$ induced from the cuspidal representation $\pi \otimes \alpha^s$. Namely, $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi \otimes \alpha^s$ is the space of all functions $\widetilde{f}_s \colon G(\mathbb{A}) \to V_\pi$ satisfying

$$\widetilde{f}_s(umg) = \alpha(m)^{s+n/2}\pi(m)(\widetilde{f}_s(g))$$

for any $u \in N(\mathbb{A})$, $m \in M(\mathbb{A})$ and $g \in G(\mathbb{A})$. By evaluating cusp form $\widetilde{f}_s(g) \in V_\pi$ at identity 1_{2n}, we obtain a complex-valued function $f_s(g) = \widetilde{f}_s(g)(1_{2n})$ in $g \in G(\mathbb{A})$. We denote by $I(\pi\otimes\alpha^{s+n/2})$ the space consisting of those functions f_s which are smooth and K-finite. Furthermore, we denote by $W(I(\pi\otimes\alpha^{s+n/2}), \psi_\Delta^{-1})$ the space spanned by all functions of the form

$$W_{f_s}(g) = \int_{\Delta(F)\backslash\Delta(\mathbb{A})} \psi_\Delta(\delta) f_s(\delta g) d\delta \quad (f_s \in I(\pi\otimes\alpha^{s+n/2})).$$

For each $f \in I(\pi)$, we define

$$E(s,f,g) = \sum_{\gamma\in P(F)\backslash G(F)} \alpha(\gamma g)^{s+n/2} f(\gamma g).$$

The series of the right-hand side is absolutely convergent for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function on the whole s-plane. The normalizing factor of $E(s,f,g)$ is given as follows. Let ${}^LM = (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \Gamma_{E/F}$ be the L-group of M. We define the representation $r_1 \colon {}^LM \to \mathrm{Aut}(M_n(\mathbb{C}))$ by

$$r_1((g_1,g_2)\rtimes\epsilon)X = g_1{}^tX{}^tg_2 \quad (X \in M_n(\mathbb{C})).$$

For $v \in \mathbf{V}_F\backslash S$, the semisimple element $\gamma'_v \in {}^LM$ corresponding to the unramified representation π_v is given by

$$\gamma'_v = \begin{cases} \left(\begin{pmatrix} \beta_{v,1} & & \\ & \ddots & \\ & & \beta_{v,n}\end{pmatrix}, 1_n\right) \rtimes \epsilon & \text{if } v \text{ remains prime in } E, \\ \left(\begin{pmatrix} \beta_{v,1} & & \\ & \ddots & \\ & & \beta_{v,n}\end{pmatrix}, \begin{pmatrix} \beta'_{v,1} & & \\ & \ddots & \\ & & \beta'_{v,n}\end{pmatrix}\right) \rtimes e & \text{if } v \text{ splits in } E. \end{cases}$$

Then we put

$$L_v(s,\pi_v,r_1) = \det(1_{n^2} - r_1(\gamma'_v)q_v^{-s})^{-1}.$$

The Euler product

$$L_S(s,\pi,r_1)=\prod_{v\notin S}L_v(s,\pi_v,r_1)$$

converges absolutely for $\mathrm{Re}(s)>>0$. It is known by [F] and [F-Z] or [Sh] that $L_S(s,\pi,r_1)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$.

Next, we define theta series on $G(\mathbb{A})$. Let Sp_{4n} be the symplectic group of size $4n$ defined over F. Then $G(\mathbb{A})$ is naturally embedded in $\mathrm{Sp}_{4n}(\mathbb{A})$. Let $\mathrm{Mp}_{4n}(\mathbb{A})\to\mathrm{Sp}_{4n}(\mathbb{A})$ be the metaplectic covering and $(\omega_\mu,\mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(\mathbb{A})))$ the Weil representation of $\mathrm{Mp}_{4n}(\mathbb{A})$ associated with μ. By [G-R], a fixed pair (μ,ν) determines a compatible splitting $s_{\mu,\nu}\colon G(\mathbb{A})\to\mathrm{Mp}_{4n}(\mathbb{A})$ of the metaplectic covering. We denote by $\omega^n_{\mu,\nu}$, or simply ω, the representation $g\mapsto\omega_\mu(s_{\mu,\nu}(g))$ of $G(\mathbb{A})$. We have precisely

$$\omega(m(a)v(b))\eta(x)=\nu(\det a)\alpha(m(a))^{1/2}\mu({}^t\overline{x}ab{}^t\overline{a}x)\eta({}^t\overline{a}x),$$

$$\omega(J_n)\eta(x)=\int_{M_{n,1}(\mathbb{A}_E)}\mu(\mathrm{Tr}_{E/F}({}^t\overline{y}x))\eta(y)dy.$$

Let $\mathcal{S}_0(R_{E/F}(M_{n,1})(\mathbb{A}))$ be the set of K-finite functions in $\mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$. For each $\eta\in\mathcal{S}_0(\mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$, the theta series θ^n_η is defined to be

$$\theta^n_\eta(g)=\sum_{x\in\mathrm{R}_{E/F}(M_{n,1})(F)}\omega(g)\eta(x).$$

If we put

$$R_\eta(g)=\omega(g)\eta(\varepsilon_n),\quad \varepsilon_n=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\in M_{n,1}(F),$$

then we have

$$\theta^n_\eta(g)=\omega(g)f(0)+\sum_{Q(F)\backslash M(F)}R_\eta(\gamma g),$$

where $Q=\{m(a)\in M\colon {}^t\overline{a}\varepsilon_n=\varepsilon_n\}$ is a subgroup of M.

We fix $\varphi\in V_\sigma$, $\varphi^*\in V_{\sigma^*}$, $f\in I(\pi)$ and $\eta\in\mathcal{S}_0(\mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$. Then we define global zeta integrals as

$$J(s,\varphi,f,\eta)=\int_{G(F)\backslash G(\mathbb{A})}\varphi(g)E(s,f,g)\theta^n_\eta(g)dg,$$

$$J^*(s,\varphi^*,f)=\int_{G(F)\backslash G(\mathbb{A})}\varphi^*(\iota(g))E(s,f,g)dg.$$

The following theorem is due to Gelbart and Piatetski-Shapiro.

THEOREM ([G-P, Theorems A and C]): *For* $\mathrm{Re}(s) >> 0$, *one has*

$$J(s,\varphi,f,\eta) = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi(g)W_f(g)R_\eta(g)\alpha(g)^{s+n/2}dg,$$
$$J^*(s,\varphi^*,f) = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} W_{\varphi^*}(\iota(g))W_f(g)\alpha(g)^{s+n/2}dg.$$

We assume that all φ, φ^*, f and η are decomposable, so that we have

$$W_\varphi(g) = \prod_{v\in\mathbf{V}_F} W_v(g_v), \quad W_{\varphi^*}(h) = \prod_{v\in\mathbf{V}_F} W_v^*(h_v),$$
$$W_f(g) = \prod_{v\in\mathbf{V}_F} W_v'(g_v), \quad R_\eta(g) = \prod_{v\in\mathbf{V}_F} R_{\eta_v}(g_v) \quad (R_{\eta_v}(g_v) = \omega_v^n(g_v)\eta_v(\varepsilon_n)).$$

Then the corresponding local integrals are given by

$$J_v(s,W_v,W_v',R_{\eta_v}) = \int_{U(F_v)\backslash G(F_v)} W_v(g)W_v'(g)R_{\eta_v}(g)\alpha_v(g)^{s+n/2}dg,$$
$$J_v^*(s,W_v^*,W_v') = \int_{U(F_v)\backslash G(F_v)} W_v^*(\iota(g))W_v'(g)\alpha_v(g)^{s+n/2}dg.$$

In the rest of this section, we assume that v splits in E. Then $E\otimes F_v \cong F_v\oplus F_v$ and a projection onto the first factor induces isomorphisms $G(F_v)\cong \mathrm{GL}_{2n}(F_v)$ and $G^*(F_v)\cong \mathrm{GL}_{2n+1}(F_v)$, so that we identify these groups. We rewrite the integrands of the local zeta integrals according to these identifications. We mainly consider the function

$$W_v(m)W_v'(m)R_{\eta_v}(m)\alpha_v(m)^{s+n/2} \quad (m\in M(F_v))$$

for $W_v\in W^\infty(\sigma_v,\psi_v)$, $W_v'\in W^\infty(I(\pi_v),\psi_{\Delta,v}^{-1})$ and $\eta_v\in \mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(F_v))$.

The groups $M(F_v)$ and $N(F_v)$ are written as

$$M(F_v) = \left\{ m(a_1,a_2) = \begin{pmatrix} a_1 & \\ & {}^t a_2{}^{-1} \end{pmatrix} : a_1,\ a_2 \in \mathrm{GL}_n(F_v)\right\},$$
$$N(F_v) = \left\{ v(b) = \begin{pmatrix} 1_n & b \\ & 1_n \end{pmatrix} : b\in M_n(F_v)\right\}.$$

The homomorphism $(a_1,a_2)\mapsto m(a_1,a_2)$ maps $\Delta_0(F_v)\times\Delta_0(F_v)$ to $\Delta(F_v)$. Notice that the maximal unipotent subgroup $U(F_v) = \Delta(F_v)\ltimes N(F_v)$ of $\mathrm{GL}_{2n}(F_v)$ is not the group of upper triangular unipotent matrices. Every $W_v\in W^\infty(\sigma_v,\psi_v)$ satisfies

$$W_v(m(\delta_1,\delta_2)v(b)g) = \psi_{0,v}(\delta_1\delta_2)\mu_v(-b_{nn})W_v(g)$$

for any $\delta_1, \delta_2 \in \Delta_0(F_v)$, $b = (b_{ij}) \in M_n(F_v)$ and $g \in \mathrm{GL}_{2n}(F_v)$. If we take

$$w_n = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in \mathrm{GL}_n(F_v),$$

then $m(1_n, w_n)U(F_v)m(1_n, w_n)^{-1}$ is the upper triangular unipotent subgroup of $\mathrm{GL}_{2n}(F_v)$, and hence the function $(W_v)^{w_n}(g) = W_v(m(1_n, w_n)g)$ becomes an ordinary Whittaker function of $\mathrm{GL}_{2n}(F_v)$, i.e. $(W_v)^{w_n}$ satisfies

$$(W_v)^{w_n}(\begin{pmatrix} 1 & u_{12} & u_{23} & \cdots & u_{1\,2n} \\ 0 & 1 & u_{23} & \cdots & u_{2\,2n} \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & 1 \end{pmatrix} g)$$
$$= \mu_v(u_{12} + \cdots + u_{n-1\,n} - u_{n\,n+1} - u_{n+1\,n+2} - \cdots - u_{2n-1\,2n})(W_v)^{w_n}(g).$$

The representation $\pi_v \circ m$ of $\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v)$ is written as $\pi_{v,1} \otimes \pi_{v,2}$, where both $\pi_{v,1}$ and $\pi_{v,2}$ are irreducible admissible representations of $\mathrm{GL}_n(F_v)$. We denote by $W^\infty(\pi_{v,i}, \psi_{0,v}^{-1})$ the Whittaker model of $\pi_{v,i}$ with respect to $\psi_{0,v}^{-1}$ for $i = 1, 2$. Since $W^\infty(\pi_v, \psi_{\Delta,v}^{-1})$ is identified with $W^\infty(\pi_{v,1}, \psi_{0,v}^{-1}) \otimes W^\infty(\pi_{v,2}, \psi_{0,v}^{-1})$, we may assume W'_v is of the form

$$W'_v(m(a_1, a_2)) = W'_{v,1}(a_1)W'_{v,2}(a_2),$$

where $W'_{v,1} \in W^\infty(\pi_{v,1}, \psi_{0,v}^{-1})$ and $W'_{v,2} \in W^\infty(\pi_{v,2}, \psi_{0,v}^{-1}))$. The character ν_v of $(E \otimes F_v)^\times$ is written as $\nu_{v,1} \otimes \nu_{v,2}$, where both $\nu_{v,1}$ and $\nu_{v,2}$ are characters of $F_v^\times$. By the assumption on ν, we have $\nu_{v,1} = \nu_{v,2}^{-1}$. Let $\alpha_{0,v}$ be the quasicharacter of $\mathrm{GL}_n(F_v)$ defined as $\alpha_{0,v}(g) = |\det g|_{F_v}$. If we take η_v of the form $\eta_{v,2} \otimes \eta_{v,1}$ ($\eta_{v,1}, \eta_{v,2} \in \mathcal{S}(M_{n,1}(F_v))$), then

$$\omega_v^n(m(a_1, a_2))\eta_v(x_1, x_2)$$
$$= \nu_{v,1}(\det a_1)\nu_{v,2}(\det a_2)\alpha_{0,v}(a_1a_2)^{1/2}\eta_{v,1}({}^t a_1 x_1)\eta_{v,2}({}^t a_2 x_2).$$

Therefore,

$$R_{\eta_v}(m(a_1, a_2)) = \nu_{v,1}(\det a_1)\nu_{v,2}(\det a_2)\alpha_{0,v}(a_1a_2)^{1/2}\eta_{v,1}({}^t a_1 \varepsilon_n)\eta_{v,2}({}^t a_2 \varepsilon_n).$$

Summing up, we have

$$\begin{aligned} & W_v(m(a_1, a_2))W'_v(m(a_1, a_2))R_{\eta_v}(m(a_1, a_2))\alpha_v(m(a_1, a_2))^{s+n/2} \\ (3.1)\quad & = W_v(m(a_1, a_2))W'_{v,1}(a_1)W'_{v,2}(a_2) \\ & \quad \times \nu_{v,1}(\det a_1)\nu_{v,2}(\det a_2)\eta_{v,1}({}^t a_1\varepsilon_n)\eta_{v,2}({}^t a_2\varepsilon_n)\alpha_{0,v}(a_1a_2)^{s+n/2+1/2}. \end{aligned}$$

In a similar fashion, we obtain for $m^*(a_1,a_2)=\iota(m(a_1,a_2))$

$$\begin{aligned}(3.2)\qquad & W_v^*(m^*(a_1,a_2))W_v'(m^*(a_1,a_2))\alpha_v(m^*(a_1,a_2)) \\ & = W_v^*(m^*(a_1,a_2))W'_{v,1}(a_1)W'_{v,2}(a_2)\alpha_{0,v}(a_1a_2)^{s+n/2+1/2}.\end{aligned}$$

We give some formula of $W_v(m(a_1,a_2))$ (resp. $W_v^*(m^*(a_1,a_2))$) when σ_v (resp. σ_v^*) is realized as a quotient of a representation induced from a representation of $M(F_v)$ (resp. $M^*(F_v)$). Let $\tau_{v,1},\tau_{v,2},\tau^*_{v,1}$ and $\tau^*_{v,2}$ be admissible representations of $\mathrm{GL}_n(F_v)$ and χ_v^* be a quasicharacter of $\mathrm{GL}_1(F_v)$. Then $\tau_v=\tau_{v,1}\otimes\tau_{v,2}$ and $\tau_v^*=\tau^*_{v,1}\otimes\chi_v^*\otimes\tau^*_{v,2}$ are regarded as representations of $M(F_v)$ and $M^*(F_v)$, respectively, by

$$\tau_v(m(a_1,a_2))=\tau_{v,1}(a_1)\otimes\tau_{v,2}(a_2),$$
$$\tau_v^*(\begin{pmatrix} a_1 & & \\ & \varepsilon & \\ & & {}^ta_2^{-1}\end{pmatrix})=\tau^*_{v,1}(a_1)\otimes\chi_v^*(\varepsilon)\otimes\tau^*_{v,2}(a_2).$$

Let ${}^tP$ (resp. ${}^tP^*$) be the parabolic subgroup opposite to P (resp. P^*). We denote by $\mathrm{Ind}_{{}^tP(F_v)}^{G(F_v)}\tau_v$ (resp. $\mathrm{Ind}_{{}^tP^*(F_v)}^{G^*(F_v)}\tau_v^*$) the representation induced from τ_v (resp. τ_v^*). Namely, $\mathrm{Ind}_{{}^tP(F_v)}^{G(F_v)}\tau_v$ consists of all smooth functions $f\colon G(F_v)\to V^\infty_{\tau_{v,1}}\otimes V^\infty_{\tau_{v,2}}$ satisfying

$$f(\begin{pmatrix} a_1 & 0\\ u & {}^ta_2^{-1}\end{pmatrix}g)=\alpha_{0,v}(a_1a_2)^{-n/2}\tau_{v,1}(a_1)\otimes\tau_{v,2}(a_2)f(g)$$

for all $a_1,a_2\in\mathrm{GL}_n(F_v)$, $u\in M_n(F_v)$ and $g\in G(F_v)$. Similarly, $\mathrm{Ind}_{{}^tP^*(F_v)}^{G^*(F_v)}\tau_v^*$ consists of all smooth functions $f^*\colon G^*(F_v)\to V^\infty_{\tau^*_{v,1}}\otimes V^\infty_{\tau^*_{v,2}}$ satisfying

$$f^*(\begin{pmatrix} a_1 & 0 & 0\\ {}^tx & \varepsilon & 0\\ u & y & {}^ta_2^{-1}\end{pmatrix}g)=\alpha_{0,v}(a_1a_2)^{-(n+1)/2}\chi_v^*(\varepsilon)\tau^*_{v,1}(a_1)\otimes\tau^*_{v,2}(a_2)f^*(g)$$

for all $a_1,a_2\in\mathrm{GL}_n(F_v)$, $\varepsilon\in\mathrm{GL}_1(F_v)$, $u\in M_n(F_v)$, $x,y\in M_{n,1}(F_v)$ and $g\in G^*(F_v)$. In the rest of this section, we assume that σ_v (resp. σ_v^*) is an irreducible quotient of $\mathrm{Ind}_{{}^tP(F_v)}^{G(F_v)}\tau_v$ (resp. $\mathrm{Ind}_{{}^tP^*(F_v)}^{G^*(F_v)}\tau_v^*$). This is always true if v is archimedean. Then τ_v (resp. τ_v^*) is generic and the Whittaker model of σ_v (resp. σ_v^*) is equal to the Whittaker model of $\mathrm{Ind}_{{}^tP(F_v)}^{G(F_v)}\tau_v$ (resp. $\mathrm{Ind}_{{}^tP^*(F_v)}^{G^*(F_v)}\tau_v^*$). For $i=1,2$, denote by $W^\infty(\tau_{v,i},\psi_{0,v})$ (resp. $W^\infty(\tau^*_{v,i},\psi_{0,v})$) the Whittaker model of $\tau_{v,i}$ (resp. $\tau^*_{v,i}$) with respect to $\psi_{0,v}$. The next Lemma follows from the same method as that of the proof of [J-S3, Lemma 10.1] and [J-P-S, Proposition (9,1)].

LEMMA 1: *The notation being as above:*

(1) *For given $W_{v,1} \in W^\infty(\tau_{v,1}, \psi_0), W_{v,2} \in W^\infty(\tau_{v,2}, \psi_0)$ and $\phi_v \in \mathcal{S}(M_n(F_v))$, there is an element $W_{v,0} \in W^\infty(\sigma_v, \psi_v)$ such that*

$$W_{v,0}(m(a_1, a_2)) = W_{v,1}(a_1)W_{v,2}(a_2)\phi_v({}^t a_2 w_n \varepsilon_{1n} a_1)\alpha_{0,v}(a_1 a_2)^{n/2},$$

where $\varepsilon_{1n} \in M_n(F_v)$ is the matrix whose $(1, n)$ component is 1 and other components are 0.

(2) *For given $W^*_{v,1} \in W^\infty(\tau^*_{v,1}, \psi_0), W^*_{v,2} \in W^\infty(\tau^*_{v,2}, \psi_0)$ and $\phi^*_{v,1}, \phi^*_{v,2} \in \mathcal{S}(M_{n,1}(F_v))$, there is an element $W^*_{v,0} \in W^\infty(\sigma^*_v, \psi_v)$ such that*

$$W^*_{v,0}(m^*(a_1, a_2)) = W^*_{v,1}(a_1)W^*_{v,2}(a_2)\phi^*_{v,1}({}^t a_1 \varepsilon_n)\phi^*_{v,2}({}^t a_2 \varepsilon_n)\alpha_{0,v}(a_1 a_2)^{(n+1)/2}.$$

4. Integral representations of L-functions

Explicit computations of $J^*_v(s, W^*_v, W'_v)$ for unramified data were accomplished by Tamir.

LEMMA 2 ([Ta]): *Let $W^*_{v,0}$ and $W'_{v,0}$ be the class 1 Whittaker functions in $W(\sigma^*_v, \psi^*_v)$ and $W(I(\pi_v), \psi^{-1}_{\Delta,v})$, respectively, normalized so that $W^*_{v,0}(1_{2n+1}) = W'_{v,0}(1_{2n}) = 1$. Then one has*

$$J^*_v(s, W^*_{v,0}, W'_{v,0}) = \frac{L_v(s + 1/2, \sigma^*_v \times \pi_v)}{L_v(2s + 1, \pi_v, r_1)}.$$

By a slight modification of Tamir's method, we can compute $J_v(s, W_v, W'_v, R_\eta)$, i.e. we obtain

LEMMA 3: *Let $W_{v,0} \in W(\sigma_v, \psi_v)$ and $W'_{v,0} \in W(I(\pi_v), \psi^{-1}_{\Delta,v})$ be the class 1 Whittaker functions, and let $\eta_{v,0}$ be the characteristic function of $M_{n,1}(\mathcal{O}_{E_v})$. Then one has*

$$J_v(s, W_{v,0}, W'_{v,0}, R_{\eta_{v,0}}) = \frac{L_v(s + 1/2, \sigma_v \times (\pi_v \otimes \nu_v))}{L_v(2s + 1, \pi_v, r_1)}.$$

Next, we calculate the local integrals for ramified primes. We assume that v is an archimedean place. Let $\mathcal{S}_0(\mathrm{R}_{E/F}(M_{n,1})(F_v))$ be the space of K_v-finite functions in $\mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(F_v))$.

LEMMA 4: *For $W_v \in W^\infty(\sigma_v, \psi_v), W^*_v \in W^\infty(\sigma^*_v, \psi^*_v), W' \in W^\infty(I(\pi_v), \psi^{-1}_{\Delta,v})$ and $\eta_v \in \mathcal{S}_0(\mathrm{R}_{E/F}(M_{n,1})(F_v))$, both $J_v(s, W_v, W'_v, R_{\eta_v})$ and $J^*_v(s, W^*_v, W'_v)$ are* *absolutely convergent for* $\mathrm{Re}(s) >> 0$ *and extend to meromorphic functions on the whole* s*-plane.*

Proof: We consider the integral $J_v(s, W_v, W_v', R_{\eta_v})$. Since η_v is K_v-finite, it is enough to determine the absolute convergence and the analytic continuation of

$$\int_{T(F_v)} W_v(t)W_v'(t)R_{\eta_v}(t)\alpha_v(t)^{s+n/2}\delta_{B(F_v)}(t)^{-1}dt, \tag{4.1}$$

where $\delta_{B(F_v)}$ denotes the modular character of $B(F_v)$.

First we assume that v remains prime in E. We denote by ω_{π_v} the central character of π_v. We put

$$a = \mathrm{diag}(t_1t_2\cdots t_n, t_2\cdots t_n, \ldots, t_n) \in \mathrm{R}_{E/F}(T_0)(F_v)$$

and $t = m(a) \in T(F_v)$. By a similar argument to [So, Proposition (3.3)] and [J-S2, Proposition 1 or 2], we know that W_v and W_v' are of the forms

$$\begin{aligned} W_v(t) &= \sum_{\alpha\in X_{\sigma_v}} \phi_\alpha(t_1,\ldots,t_n)\alpha(t_1,\ldots,t_n), \\ W_v'(t) &= \omega_{\pi_v}(t_n)\sum_{\beta\in X_{\pi_v}} \phi_\beta(t_1,\ldots,t_{n-1})\beta(t_1,\ldots,t_{n-1}), \end{aligned} \tag{4.2}$$

where X_{σ_v} (resp. X_{π_v}) is a finite set of finite functions α on $(E_v^\times)^n$ (resp. β on $(E_v^\times)^{n-1}$) and ϕ_α (resp. ϕ_β) are Schwartz–Bruhat functions on $(E_v)^n$ (resp. $(E_v)^{n-1}$). Furthermore, $R_{\eta_v}(t)$ is of the form

$$\eta_v(\bar{t}_n\varepsilon_n)\nu_v(t_1t_2^2\cdots t_n^n)|t_1t_2^2\cdots t_n^n|_{E_v}^{1/2}.$$

Thus (4.1) is a linear combination of integrals of the form

$$\int_{(E_v^\times)^n} \phi_\alpha(t_1,\ldots,t_n)\alpha(t_1,\ldots,t_n)\phi_\beta(t_1,\ldots,t_{n-1})\beta(t_1,\ldots,t_{n-1})\eta_v(\bar{t}_n\varepsilon_n)$$
$$\omega_{\pi_v}(t_n)\prod_{j=1}^{n}\nu_v(t_j)^j|t_j|_{E_v}^{j(s+n/2)+j^2-2nj+j/2}d^\times t_1\cdots d^\times t_n.$$

It is well known that this integral converges absolutely for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function.

Next we assume that v splits in E. By (3.1), the integral (4.1) is equal to

$$\begin{aligned}\int_{T_0(F_v)\times T_0(F_v)} & W_v(m(a_1,a_2))W_{v,1}'(a_1)W_{v,2}'(a_2)\nu_{v,1}(\det a_1)\nu_{v,2}(\det a_2) \\ & \times \eta_{v,1}({}^ta_1\varepsilon_n)\eta_{v,2}({}^ta_2\varepsilon_n)\alpha_{0,v}(a_1a_2)^{s+n/2+1/2}\delta_{B_0(F_v)}(a_1a_2)^{-1}da_1da_2,\end{aligned} \tag{4.3}$$

where $\delta_{B_0(F_v)}$ denotes the modular character of $B_0(F_v)$. We put

$$a_i = \mathrm{diag}(t_{i1}t_{i2}\cdots t_{in}, t_{i2}\cdots t_{in}, \ldots, t_{in}) \in T_0(F_v) \quad (i=1,2).$$

Let $\omega_{\pi_{v,i}}$ be the central character of $\pi_{v,i}$ for $i=1,2$. By [J-S, Proposition 1], $W'_{v,i}(a_i)$ is of the form

$$W'_{v,i}(a_i) = \omega_{\pi_{v,i}}(t_{in}) \sum_{\beta^{(i)} \in X_{\pi_{v,i}}} \phi_{\beta^{(i)}}(t_{i1}, \ldots, t_{i\,n-1})\beta^{(i)}(t_{i1}, \ldots, t_{i\,n-1})$$

for $i=1,2$. Here $X_{\pi_{v,i}}$ is a finite set of finite functions $\beta^{(i)}$ on $(F_v^\times)^{n-1}$ and $\phi_{\beta^{(i)}}$ are Schwartz–Bruhat functions on $(F_v)^{n-1}$. We evaluate $W_v(m(a_1,a_2))$ by applying [J-S, Proposition 1] to $(W_v)^{w_n}(m(a_1,a_2))$. Then $W_v(m(a_1,a_2))$ is of the form

$$\begin{aligned}&\omega_{\sigma_v}(t_{21}\cdots t_{2n})^{-1} \sum_{\alpha \in X_{\sigma_v}} \phi_\alpha(t_{11}, \ldots, t_{1\,n-1}, t_{1n}t_{2n}, t_{2\,n-1}, \ldots, t_{21})\\ &\times \alpha(t_{11}, \ldots, t_{1\,n-1}, t_{1n}t_{2n}, t_{2\,n-1}, \ldots, t_{21}).\end{aligned}$$

Here ω_{σ_v} denotes the central character of σ_v, X_{σ_v} is a finite set of finite functions α on $(F_v^\times)^{2n-1}$ and ϕ_α are Schwartz–Bruhat functions on $(F_v)^{2n-1}$. Therefore, (4.3) is a finite linear combination of integrals of the form

$$\begin{aligned}&\int_{(F_v^\times)^{2n}} \phi_\alpha(t_{11}, \ldots, t_{1\,n-1}, t_{1n}t_{2n}, t_{2\,n-1}, \ldots, t_{21})\\ &\times \alpha(t_{11}, \ldots, t_{1\,n-1}, t_{1n}t_{2n}, t_{2\,n-1}, \ldots, t_{21})\omega_{\sigma_v}(t_{21}\cdots t_{2n})^{-1}\\ &\times \left\{\prod_{i=1}^{2} \omega_{\pi_{v,i}}(t_{in})\phi_{\beta^{(i)}}(t_{i1}, \ldots, t_{i\,n-1})\beta^{(i)}(t_{i1}, \ldots, t_{i\,n-1})\eta_{v,i}(t_{in}\varepsilon_n)\right\}\\ &\times \left\{\prod_{i=1}^{2}\prod_{j=1}^{n} \nu_{v,i}(t_{ij})^j |t_{ij}|_{F_v}^{(s+n/2+1/2)j-(n-j)j}\right\} d^\times t_{11}\cdots d^\times t_{2n}\,.\end{aligned} \tag{4.4}$$

We change the variable t_{1n} to $t_{1n}t_{2n}^{-1}$. Then the integral in the variable t_{2n} is equal to

$$\int_{F_v^\times} \omega_{\sigma_v}(t_{2n})^{-1}\omega_{\pi_{v,1}}(t_{2n})^{-1}\omega_{\pi_{v,2}}(t_{2n})\eta_{v,1}(t_{1n}t_{2n}^{-1}\varepsilon_n)\eta_{v,2}(t_{2n}\varepsilon_n)d^\times t_{2n}.$$

If we denote this by $\phi(t_{1n})$, then ϕ is regarded as a Schwartz–Bruhat function on F_v. Thus (4.4) is equal to

$$\int_{(F_v^\times)^{2n-1}} \phi_\alpha(t_{11},\dots,t_{1\,n-1},t_{1n},t_{2\,n-1},\dots,t_{21})$$
$$\times\, \alpha(t_{11},\dots,t_{1\,n-1},t_{1n},t_{2\,n-1},\dots,t_{21})\phi(t_{1n})\omega_{\sigma_v}(t_{21}\cdots t_{2\,n-1})^{-1}$$
$$\times \left\{\prod_{i=1}^{2}\omega_{\pi_{v,i}}(t_{in})\phi_{\beta^{(i)}}(t_{i1},\dots,t_{i\,n-1})\beta^{(i)}(t_{i1},\dots,t_{i\,n-1})\right\}$$
$$\times \left\{\prod_{i=1}^{2}\prod_{j=1}^{n-1}\nu_{v,i}(t_{ij})^j|t_{ij}|_{F_v}^{(s+n/2+1/2)j-(n-j)j}\right\}$$
$$\times\, \nu_{v,1}(t_{1n})^n|t_{1n}|_{F_v}^{(s+n/2+1/2)n}d^\times t_{11}\cdots d^\times t_{2\,n-1}.$$

This integral converges absolutely for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function on $\mathbb{C}$.

In a similar fashion, we can prove the absolute convergence and the analytic continuation of the integral $J_v^*(s,W_v^*,W_v')$ (cf. [So, Section 5]). ∎

LEMMA 5: *There exist $W_{v,0}\in W(\sigma_v,\psi_v)$, $W'_{v,0}\in W(I(\pi_v),\psi_{\Delta,v}^{-1})$ and $\eta_{v,0}\in \mathcal{S}_0(\mathrm{R}_{E/F}(M_{n,1})(F_v))$ such that $J_v(s,W_{v,0},W'_{v,0},\eta_{v,0})\neq 0$ for a given $s\in\mathbb{C}$ with $\mathrm{Re}(s)>>0$.*

Proof: First, we assume that v remains prime in E. Let $K_{1,v}$ be the standard maximal compact subgroup of $\mathrm{GL}_n(E_v)$. We put

$$K_{M,v}=M(F_v)\cap K_v=\{m(k_1)\colon k_1\in K_{1,v}\},$$
$$Z_v^+=\{m(r1_n)\colon r>0\}.$$

Since $M(F_v)=Q(F_v)Z_v^+K_{M,v}$, $J_v(s,W_v,W_v',R_{\eta_v})$ equals

$$\int_{\Delta(F_v)\backslash M(F_v)K_v} W_v(mk)W_v'(mk)R_{\eta_v}(mk)\alpha_v(mk)^{s-n/2}dmdk$$
$$=\int_{K_{M,v}\backslash K_v}\int_{Z_v^+\times(Q(F_v)\cap K_{M,v}\backslash K_{M,v})} R_{\eta_v}(zkk')\alpha_v(z)^{s-n/2}$$
$$\times\Big\{\int_{\Delta(F_v)\backslash Q(F_v)} W_v(pzkk')W_v'(pzkk')\alpha_v(p)^{s-n/2}dp\Big\}dzdkdk'.$$

We take a $W_{v,0}\in W(\sigma_v,\psi_v)$ such that $W_{v,0}(1_{2n})\neq 0$. It follows from (4.2) that for $\mathrm{Re}(s)>>0$, $\alpha_v^sW_{v,0}$ is square integrable on $\Delta(F_v)\backslash Q(F_v)$. In other words, $\alpha_v^sW_{v,0}$ is contained in the L^2-induction space $L^2\text{-}\mathrm{Ind}_{\Delta(F_v)}^{Q(F_v)}\psi_{\Delta,v}$. On the other hand, by [J-S1, Proposition 3.8], the space $\{\alpha_v^{r_\pi} W'_v|_{Q(F_v)} : W'_v \in W(I(\pi_v), \psi_{\Delta,v}^{-1})\}$ is dense in $L^2\text{-Ind}_{\Delta(F_v)}^{Q(F_v)} \psi_{\Delta,v}^{-1}$. Thus there exists $W'_{v,0} \in W(I(\pi_v), \psi_{\Delta,v}^{-1})$ such that

$$\int_{\Delta(F_v)\backslash Q(F_v)} W_{v,0}(p) W'_{v,0}(p) \alpha_v(p)^{s-n/2} dp \neq 0$$

for a given s with $\mathrm{Re}(s) >> 0$. We put

$$\Psi'_s(W_{v,0}, W'_{v,0}; zk) = \int_{\Delta(F_v)\backslash Q(F_v)} W_{v,0}(pzk) W'_{v,0}(pzk) \alpha_v(pz)^{s-n/2} dp.$$

For $\beta \in C_0^\infty(\mathbb{R}_+^\times)$ and $\xi \in C^\infty(Q_0(E_v) \cap K_{1,v}\backslash K_{1,v})$, define the function $\eta_{\beta\otimes\xi} \in \mathcal{S}(M_{n,1}(E_v))$ by

$$\eta_{\beta\otimes\xi}(x) = \begin{cases} \beta(r)\xi(k_1) & \text{if } x = {}^t\overline{k}_1 r \varepsilon_n \in K_{1,v}\mathbb{R}_+^\times \varepsilon_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} &\int_{Z_v^+ \times (Q(F_v)\cap K_{M,v}\backslash K_{M,v})} \Psi'_s(W_{v,0}, W'_{v,0}; zk) R_{\eta_{\beta\otimes\xi}}(zk) dz dk \\ &= \int_{\mathbb{R}_+^\times \times (Q_0(E_v)\cap K_{1,v}\backslash K_{1,v})} \Psi'_s(W_{v,0}, W'_{v,0}; m(r1_n)m(k_1))\beta(r)\xi(k_1) d^\times r dk_1. \end{aligned}$$

Since $\Psi'_s(W_{v,0}, W'_{v,0}; 1_{2n}) \neq 0$, we can take β and ξ such that the last integral is nonzero. Therefore, there exists $\eta_{v,0} \in \mathcal{S}_0(M_{n,1}(E_v))$ such that

$$\int_{Z_v^+ \times (Q(F_v)\cap K_{M,v}\backslash K_{M,v})} \Psi'_s(W_{v,0}, W'_{v,0}; zk) R_{\eta_{v,0}}(zk) dz dk \neq 0.$$

We put

$$\begin{aligned} &\Psi_s(W_{v,0}, W'_{v,0}, R_{\eta_{v,0}}; k') \\ &= \int_{Z_v^+ \times (Q(F_v)\cap K_{M,v}\backslash K_{M,v})} \Psi'_s(W_{v,0}, W'_{v,0}; zkk') R_{\eta_{v,0}}(zkk') dz dk. \end{aligned}$$

Let $C_{K_v}^\infty(K_{M,v}\backslash K_v)$ be the space consisting of smooth, right K_v-finite and left $K_{M,v}$-invariant functions on K_v. For $\xi' \in C_{K_v}^\infty(K_{M,v}\backslash K_v)$, define the function $W'_{v,0} \otimes \xi'$ on $G(F_v)$ by

$$W'_{v,0} \otimes \xi'(umk') = W'_{v,0}(umk')\xi'(k') \quad (u \in N(F_v),\ m \in M(F_v),\ k' \in K_v).$$

Then $W'_{v,0} \otimes \xi' \in W(I(\pi_v), \psi_{\Delta,v}^{-1})$ and

$$\int_{K_{M_v}\backslash K_v} \Psi_s(W_{v,0}, W'_{v,0} \otimes \xi', R_{\eta_{v,0}}; k')dk' = \int_{K_{M_v}\backslash K_v} \Psi_s(W_{v,0}, W'_{v,0}, R_{\eta_{v,0}}; k')\xi'(k')dk'.$$

If ξ' is taken so that the last integral is nonzero, then we have $J_v(s, W_{v,0}, W'_{v,0} \otimes \xi', R_{\eta_{v,0}}) \neq 0$.

Next, we assume that v splits in E. If we can prove that there exist $W_{v,0} \in W(\sigma_v, \psi_v)$ and $W'_{v,0} \in W(I(\pi_v), \psi_{\Delta,v}^{-1})$ such that

$$\int_{\Delta(F_v)\backslash Q(F_v)} W_{v,0}(p)W'_{v,0}(p)\alpha_v(p)^{s-n/2}dp \neq 0, \tag{4.5}$$

then we can use the same argument as that of the previous paragraph to prove the assertion. Following the convention of Section 3, we identify $G(F_v)$ with $\mathrm{GL}_{2n}(F_v)$ and π_v with $\pi_{v,1} \otimes \pi_{v,2}$. Since σ_v is irreducibe and generic, it is realized as an irreducible quotient representation of an induced representation $\mathrm{Ind}_{{}^tP(F_v)}^{G(F_v)} \tau_{v,1} \otimes \tau_{v,2}$, where $\tau_{v,1}$ and $\tau_{v,2}$ are generic representations of $\mathrm{GL}_n(F_v)$. By Lemma 1 (1), for given $W_{v,1} \in W(\tau_{v,1}, \psi_{0,v})$, $W_{v,2} \in W(\tau_{v,2}, \psi_{0,v})$ and $\phi_v \in \mathcal{S}(M_n(F_v))$, there is an element $W_{v,0} \in W^\infty(\sigma_v, \psi_v)$ such that

$$W_{v,0}(m(a_1, a_2)) = W_{v,1}(a_1)W_{v,2}(a_2)\phi({}^ta_2 w_n \varepsilon_{1\,n} a_1)\alpha_{0,v}(a_1a_2)^{n/2}.$$

If $W_{v,1}$, $W_{v,2}$ and ϕ_v are taken as $W_{v,1}(1_n) \neq 0$, $W_{v,2}(1_n) \neq 0$ and $\phi_v(w_n\varepsilon_{1\,n}) = 1$, then we have

$$W_{v,0}(m(a_1, a_2)) = W_{v,1}(a_1)W_{v,2}(a_2)\alpha_{0,v}(a_1a_2)^{n/2} \quad (a_1, a_2 \in Q_0(F_v)).$$

For $i = 1, 2$ and $\mathrm{Re}(s) >> 0$, $\alpha_{0,v}^s W_{v,i}|_{Q_0(F_v)}$ is contained in the space L^2-$\mathrm{Ind}_{\Delta_0(F_v)}^{Q_0(F_v)} \psi_{0,v}$. Since the space $\{\alpha_{0,v}^{r_\pi} W'_{v,i}|_{Q_0(F_v)} : W'_{v,i} \in W(\pi_{v,i}, \psi_{0,v}^{-1})\}$ is dense in L^2-$\mathrm{Ind}_{\Delta_0(F_v)}^{Q_0(F_v)} \psi_{0,v}^{-1}$, there is a $W'_{v,i} \in W(\pi_{v,i}, \psi_{0,v}^{-1})$ such that

$$\int_{\Delta_0(F_v)\backslash Q_0(F_v)} W_{v,i}(p')W'_{v,i}(p')\alpha_{0,v}(p')^{s-n/2}dp' \neq 0$$

for a given $s \in \mathbb{C}$ with $\mathrm{Re}(s) >> 0$ and $i = 1, 2$. If we set $W'_{v,0} = W'_{v,1} \otimes W'_{v,2}$, then we have (4.5). Since $W'_{v,0}$ is taken as K_v-finite, $W_{v,0}$ can be also taken in $W(\sigma_v, \psi_v)$. ∎

In the case of G^*, we can use an argument of [So, Proposition (7.2)]. Since v is archimedean and π_v is generic, there is a generic admissible representation τ_v^* of $M^*(F_v)$ such that σ_v^* is realized as a quotient representation of $\mathrm{Ind}_{{}^tP^*(F_v)}^{G^*(F_v)}\tau_v^*$. Let $W^\infty(\tau_v^*,\psi_v^*)$ be the Whittaker model of τ_v^* with respect to ψ_v^*. For $W_v^{\tau^*}\in W^\infty(\tau_v^*,\psi_v^*)$, $W_v'\in W^\infty(\pi_v,\psi_{\Delta,v}^{-1})$ and $\phi_v^*\in\mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(F_v))$, we put

$$A^*(s,W_v^{\tau^*},W_v') = \int_{\mathrm{R}_{E/F}(\Delta_0)(F_v)\backslash \mathrm{R}_{E/F}(\mathrm{GL}_n)(F_v)} W_v^{\tau^*}(m(a))W_v'(m(a))\phi_v^*({}^ta\varepsilon_n)\alpha_v(m(a))^{s+(n+1)/2}da.$$

In the same way as [So, Proposition (7.2)], we obtain the following:

LEMMA 6: *For given $W_v^{\tau^*}\in W^\infty(\tau_v^*,\psi_v^*)$, $W_v'\in W^\infty(\pi,\psi_{\Delta,v}^{-1})$, and $\phi_v^*\in\mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(F_v))$, there are $W_v^{*(j)}\in W^\infty(\sigma_v^*,\psi_v^*)$ and $W_v'^{(j)}\in W^\infty(I(\pi_v),\psi_{\Delta,v}^{-1})$, $1\le j\le \ell$, such that*

$$\sum_{j=1}^{\ell} J_v^*(s,W_v^{*(j)},W_v'^{(j)}) = A^*(s,W_v^{\tau^*},W_v').$$

In particular, for a given $s\in\mathbb{C}$, there are $W_{v,0}^*\in W^\infty(\sigma_v^*,\psi_v^*)$ and $W_{v,0}'\in W^\infty(I(\pi_v),\psi_{\Delta,v}^{-1})$ such that $J_v^*(s,W_{v,0}^*,W_{v,0}')\neq 0$. If $\mathrm{Re}(s)>>0$, we can take $W_{v,0}^*$ and $W_{v,0}'$ as elements in $W(\sigma_v^*,\psi_v^*)$ and $W(I(\pi_v),\psi_{\Delta,v}^{-1})$, respectively.

When $n=1$, Koseki and Oda have explicitly computed $J_v^*(s,W_v^*,W_v')$ provided that σ_v^* belongs to the large discrete series. Their result states that the "g.c.d." of the integrals $J_v^*(s,W_v^*,W_v')$ turns out to be a product of three gamma functions ([K-O, Theorem (6.8)]).

Finally we calculate finite ramified local integrals. Let v be a finite prime. The following Lemma is obtained by an analogous calculation as in the proof of Lemma 4.

LEMMA 7: *Both integrals $J_v(s,W_v,W_v',R_{\eta_v})$ and $J_v^*(s,W_v^*,W_v')$ are absolutely convergent for $\mathrm{Re}(s)>>0$ and become rational functions of q_v^{-s}.*

We prove that both $J_v(s,W_v,W_v',R_{\eta_v})$ and $J_v^*(s,W_v^*,W_v')$ can be made constant for some W_v,W_v^*,W_v' and η_v. We need some Lemma when v splits in E. In this case, let $\ell=2n$ or $2n+1$. For $0\le j\le n$, P_j stands for the standard upper triangular parabolic subgroup of GL_ℓ with Levi factor $\mathrm{GL}_j\times(\mathrm{GL}_1)^{\ell-2j}\times\mathrm{GL}_j$. Let N_j be the unipotent radical of P_j, which is written as

$$N_j=\left\{u_j(\delta,x,y,z)=\begin{pmatrix}1_j & x & z\\ 0 & \delta & {}^ty\\ 0 & 0 & 1_j\end{pmatrix} : \delta\in U_{\ell-2j},\ \ x,y\in M_{j,\ell-2j},\ \ z\in M_j\right\}.$$

Here $U_{\ell-2j}$ denotes the maximal upper triangular unipotent subgroup of $\mathrm{GL}_{\ell-2j}$. Let ψ_ℓ be the character of $N_\ell(F_v) = U_\ell(F_v)$ defined by

$$\psi_\ell((u_{ij})) = \begin{cases} \mu_v(u_{12} + \cdots + u_{n-1\,n} - u_{n\,n+1} - \cdots - u_{2n-1\,2n}) & \text{if } \ell = 2n, \\ \mu_v(u_{12} + \cdots + u_{n\,n+1} - u_{n+1\,n+2} - \cdots - u_{2n\,2n+1}) & \text{if } \ell = 2n+1. \end{cases}$$

Since P_j contains U_ℓ, we can consider the representation of $P_j(F_v)$ induced from ψ_ℓ. Thus we denote by $\mathrm{Ind}_{U_\ell(F_v)}^{P_j(F_v)} \psi_\ell$ (resp. $c\text{-}\mathrm{Ind}_{U_\ell(F_v)}^{P_j(F_v)} \psi_\ell$) the space of all locally constant functions (resp. locally constant and compactly supported modulo $U_\ell(F_v)$ functions) $\phi\colon P_j(F_v) \to \mathbb{C}$ which satisfies $\phi(up) = \psi_\ell(u)\phi(p)$ for any $u \in U_\ell(F_v)$ and $p \in P_j(F_v)$. The following Lemma is proved by the same method as [G-K, Proposition 2].

LEMMA 8: *Let $1 \leq j \leq n$. If V is a nonzero $P_j(F_v)$-invariant subspace of $\mathrm{Ind}_{U_\ell(F_v)}^{P_j(F_v)} \psi_\ell$, then V contains the space $c\text{-}\mathrm{Ind}_{U_\ell(F_v)}^{P_j(F_v)} \psi_\ell$.*

By using this Lemma when v splits in E and the standard argument (cf. [G-P, Proposition (12.4)]), we obtain the following:

LEMMA 9: *Let v be any finite place.*
(1) *There are $W_{v,0}$, $W'_{v,0}$ and $\eta_{v,0}$ such that $J_v(s, W_{v,0}, W'_{v,0}, R_{\eta_{v,0}}) = 1$.*
(2) *There are $W^*_{v,0}$ and $W'_{v,0}$ such that $J^*_v(s, W^*_{v,0}, W'_{v,0}) = 1$.*

Summing up Lemmas 2–9, we obtain the following:

PROPOSITION: *For an appropriate choice of $\varphi_0 \in V_\sigma$, $f_0 \in I(\pi)$ and $\eta_0 \in \mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$, one has*

$$J(s, \varphi_0, f_0, \eta_0) = \left(\prod_{v \in S_\infty} J_v(s, W_{v,0}, W'_{v,0}, R_{\eta_{v,0}})\right) \frac{L_S(s+1/2, \sigma \times (\pi \otimes \nu))}{L_S(2s+1, \pi, r_1)},$$

where S_∞ is the subset consisting of infinite places in S. Similarly, for an appropriate choice of $\varphi_0^* \in V_{\sigma^*}$, and $f_0 \in I(\pi)$, one has

$$J^*(s, \varphi_0^*, f_0) = \left(\prod_{v \in S_\infty} J^*_v(s, W^*_{v,0}, W'_{v,0})\right) \frac{L_S(s+1/2, \sigma \times \pi)}{L_S(2s+1, \pi, r_1)}.$$

5. A comparison of L-functions in a theta series lifting

Define the unitary group $H = U(n(2n+1), n(2n+1))$ by the group consisting of elements $h \in \mathrm{R}_{E/F}(\mathrm{GL}_{2n(2n+1)})$ satisfying

$${}^t h \begin{pmatrix} 0 & 1_{n(2n+1)} \\ -1_{n(2n+1)} & 0 \end{pmatrix} \bar{h} = \begin{pmatrix} 0 & 1_{n(2n+1)} \\ -1_{n(2n+1)} & 0 \end{pmatrix}.$$

We define the Weil representation $\omega' = \omega_{\mu,\nu}^{n(2n+1)}$ of $H(\mathbb{A})$ acting on the space $\mathcal{S}(\mathrm{R}_{E/F}(M_{n(2n+1)})(\mathbb{A}))$ the same way as in Section 3. Since $G^*(\mathbb{A})\times G(\mathbb{A})$ modulo $\{(t1_{2n+1}, \bar{t}1_{2n})\colon t \in \mathbb{A}_E^\times,\ \ t\bar{t} = 1\}$ is embedded in $H(\mathbb{A})$, the representation ω' can be restricted to $G^*(\mathbb{A}) \times G(\mathbb{A})$. We identify $\mathcal{S}(\mathrm{R}_{E/F}(M_{n(2n+1)})(\mathbb{A}))$ with $\mathcal{S}(\mathrm{R}_{E/F}(M_{2n,n})(\mathbb{A}) \oplus \mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$. If η is of the form $\eta = \eta_1 \otimes \eta_2$, $\eta_1 \in \mathcal{S}(\mathrm{R}_{E/F}(M_{2n,n})(\mathbb{A})), \eta_2 \in \mathcal{S}(\mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$, then the action of $M^*(\mathbb{A}) \times G(\mathbb{A})$ to η is described as follows (cf. [Wal]):

$$\begin{aligned}(5.1)\qquad &\omega'(m^*(a,\varepsilon),g)\eta(x,y)\\ &= \nu(\varepsilon)^n\nu(\det a)^{2n}\nu(\det g)^n|\det a|_{\mathbb{A}_E}^n\eta_1(g^{-1}x\bar{a})\omega_{\mu,\nu}^n(g)\eta_2(\bar{\varepsilon}y).\end{aligned}$$

For $\varphi \in V_\sigma$ and $\eta \in \mathcal{S}(\mathrm{R}_{E/F}(M_{2n,n})(\mathbb{A}) \oplus \mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))$, the theta series lift φ_η^* is defined to be

$$\varphi_\eta^*(h) = \int_{G(F)\backslash G(\mathbb{A})} \nu(\det gh)^{-n}\varphi(g) \sum_{\substack{x\in \mathrm{R}_{E/F}(M_{2n,n})(F)\\ y\in \mathrm{R}_{E/F}(M_{n,1})(F)}} \omega'(h,g)\eta(x,y)dg.$$

We put

$$\Theta_{\mu,\nu}^n(\sigma) = \{\varphi_\eta^*\colon \varphi \in V_\sigma,\ \ \eta \in \mathcal{S}(\mathrm{R}_{E/F}(M_{2n,n})(\mathbb{A}) \oplus \mathrm{R}_{E/F}(M_{n,1})(\mathbb{A}))\}.$$

By assumption $W(\sigma,\psi) \neq 0$ and [Wal, Theorem 5.6] it is known that $\Theta_{\mu,\nu}^n(\sigma)$ is a nonzero automorphic representation of $G^*(\mathbb{A})$ and its ψ^*-Whittaker model is nonzero. The cuspidality criterion of $\Theta_{\mu,\nu}^n(\sigma)$ was given in [Wal, Theorem 5.3]. Notice that the definition of $\Theta_{\mu,\nu}^n$ is slightly different from the definition of ${}^1\Theta^n$ in [Wal]. There is a relation $\Theta_{\mu,\nu}^n(\sigma) = {}^1\Theta^n(\sigma\otimes\nu^{-n})\otimes\nu^{-n}$. Therefore, $\Theta_{\mu,\nu}^n(\sigma)$ is cuspidal if and only if the analogous lifting $\Theta_{\mu,\nu}^{n-1}(\sigma)$ of σ to the space of automorphic forms on the unitary group $U(n-1,n)(\mathbb{A})$ vanishes. Here we correct some misprints in the statement of [Wal, Theorem 5.3 (ii)]. The correct statement is the following: "Assume $n = 1$. Then ${}^1\Theta^1(\pi)$ is cuspidal if and only if $\pi \otimes \nu \circ \det$ is orthogonal to $\Theta^m(\mu^{-1}, \nu^{-1})$". Unfortunately, the irreducibility of $\Theta_{\mu,\nu}^n(\sigma)$ is unknown in general.

In the following, we compute $L_S(s, \Theta_{\mu,\nu}^n(\sigma) \times \pi)$ provided that $\Theta_{\mu,\nu}^n(\sigma)$ is cuspidal and irreducible. To mention a statement, let $\overline{\pi}$ be the cuspidal representation of $M(\mathbb{A})$ given by $\overline{\pi}(m) = \pi(\overline{m})$ for $m \in M(\mathbb{A})$. The Whittaker model $W(\overline{\pi}, \psi_\Delta^{-1})$ is equal to the space $\{\overline{W}'\colon W' \in W(\pi, \psi_\Delta^{-1})\}$, where $\overline{W}'$ is defined to be $\overline{W}'(m) = W'(\overline{m})$. Let $L(s,\overline{\pi}) = \prod_w L_w(s,\overline{\pi}_w)$ be the standard L-function of $\overline{\pi}$ as a cupidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$, where w runs over all places of E. For convenience, if $v \in \mathbf{V}_F$ is a finite place such that $\overline{\pi}_v$ is unramified, then we put

$$L_v(s,\overline{\pi}_v) = \begin{cases} L_w(s,\overline{\pi}_w) & \text{if } v = w \text{ on } E,\\ L_{w_1}(s,\overline{\pi}_{w_1})L_{w_2}(s,\overline{\pi}_{w_2}) & \text{if } v = w_1w_2 \text{ on } E.\end{cases}$$

THEOREM: *Assume that $\sigma^* = \Theta_{\mu,\nu}(\sigma)$ is cuspidal and irreducible. Then one has*

$$L_S(s,\sigma^* \times \pi) = L_S(s,\overline{\pi})L_S(s,\sigma \times (\overline{\pi} \otimes \nu)).$$

Proof: We recall a relation between the Whittaker functions W_φ and $W_{\varphi_\eta^*}$ (cf. [Wa1]). Put

$$\Psi(\eta)(g) = \int_{\Delta^*(\mathbb{A})} \psi^*(\delta)^{-1}\omega'(\delta,g)\eta(x_0,y_0)d\delta, \quad \begin{array}{l} x_0 = \begin{pmatrix} 1_n \\ 0 \end{pmatrix} \in M_{2n,n}(F), \\ y_0 = \varepsilon_n \in M_{n,1}(F). \end{array}$$

Then we have

$$W_{\varphi_\eta^*}(h) = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} \nu(\det gh)^{-n} W_\varphi(g)\Psi(\omega'(h,1_{2n})\eta)(g)dg. \tag{5.2}$$

Let τ be an irreducible finite dimensional representation of the maximal compact subgroup K and ξ_τ be the corresponding elementary idempotent. For $\Phi \in V_\pi$, we define $f_\Phi^\tau \in I(\pi)$ by

$$f_\Phi^\tau(umk) = \int_{K\cap M(\mathbb{A})} \Phi(mk')\xi_\tau(k^{-1}k')dk' \quad (u \in N(\mathbb{A}), m \in M(\mathbb{A}), k \in K).$$

If Φ is nonzero, then f_Φ^τ is nonzero for an appropriate τ. By the basic identity and (5.2), we have for $\mathrm{Re}(s) >> 0$

$$\begin{aligned} J^*(s,\varphi_\eta^*, f_\Phi^\tau) = &\int_{U(\mathbb{A})\backslash G(\mathbb{A})} W_\phi(g) \\ &\times \int_{M(\mathbb{A})} \nu(\det gm)^{-n}\omega'(\iota(m),g)\eta^\tau(x_0,y_0)W_\Phi(m)\alpha(m)^{s-n/2}dmdg, \end{aligned}$$

where we put

$$\eta^\tau(x_0,y_0) = \int_K \nu(\det k)^{-n}\xi_\tau(k^{-1})\omega'(\iota(k),1_{2n})\eta(x_0,y_0)dk.$$

We assume that η^τ is of the form $\eta^\tau = \eta_1 \otimes \eta_2$. It follows from (5.1) that

$$\nu(\det gm(a))^{-n}\omega'(\iota(m(a)),g)\eta^\tau(x_0,y_0) = |\det a|_{\mathbb{A}_E}^n \eta_1(g^{-1}x_0\overline{a})\omega(g)\eta_2(y_0).$$

Here note that $\nu(\det m(a)) = \nu(\det a)^2$. Define

$$V^s_{(\overline{W}_\Phi,\eta_1)}(g) = \alpha(g)^{-s-n/2}\int_{\mathrm{GL}_n(\mathbb{A}_E)} \eta_1(g^{-1}x_0a)\overline{W}_\Phi(m(a))|\det a|_{\mathbb{A}_E}^{s+n/2}da.$$

It is known by [Wa2] that the integral on the right hand side converges absolutely for $\mathrm{Re}(s) >> 0$ and extends to an entire function on the whole s-plane, and furthermore, $V^s_{(\overline{W}_\Phi,\eta_1)}$ is contained in $I(\overline{\pi},\psi_\Delta^{-1})$ for all s. Therefore, for $\mathrm{Re}(s) >> 0$, $J^*(s,\varphi^*_\eta,f^\tau_\Phi)$ is written as

$$J^*(s,\varphi^*_\eta,f^\tau_\Phi)=\int_{U(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi(g)V^s_{(\overline{W}_\Phi,\eta_1)}(g)R_{\eta_2}(g)\alpha(g)^{s+n/2}dg. \tag{5.3}$$

We further assume that all data are decomposable, i.e.

$$W_\varphi(g)=\prod_{v\in\mathbf{V}_F}W_v(g_v),\quad W_\Phi(m(a))=\prod_{v\in\mathbf{V}_F}W'_v(m(a_v)),$$
$$\eta_1(x)=\prod_{v\in\mathbf{V}_F}\eta_{1,v}(x_v),\quad \eta_2(y)=\prod_{v\in\mathbf{V}_F}\eta_{2,v}(y_v).$$

Then $W_{\varphi^*_\eta}$ and $W_{f^\tau_\Phi}$ are also decompsable, so that they are of the form

$$W_{\varphi^*_\eta}(h)=\prod_{v\in\mathbf{V}_F}W^*_v(h_v),\quad W_{f^\eta_\Phi}(g)=\prod_{v\in\mathbf{V}_F}W''_v(g_v).$$

If we define the local integral of $V^s_{(\overline{W}_\Phi,\eta_1)}$ by

$$\alpha_v(g)^{-s-n/2}\int_{\mathrm{R}_{E/F}(\mathrm{GL}_n)(F_v)}\eta_{1,v}(g^{-1}x_0a)\overline{W}'_v(m(a))|N_{E/F}(\det a)|_{F_v}^{s+n/2}da$$

then (5.3) decomposes to Euler products

$$J^*(s,\varphi^*_\eta,f^\tau_\Phi)=\prod_{v\in\mathbf{V}_F}J^*_v(s,W^*_v,W''_v)=\prod_{v\in\mathbf{V}_F}J_v(s,W_v,V^s_{(\overline{W}'_v,\eta_{1,v})},R_{\eta_{2,v}}).$$

By an analogous calculation as above, we have

$$J^*_v(s,W^*_v,W''_v)=J_v(s,W_v,V^s_{(\overline{W}'_v,\eta_{1,v})},R_{\eta_{2,v}})$$

for each $v\in\mathbf{V}_F$. We compute the unramified factors. For $v\in\mathbf{V}_F(\sigma,\sigma^*,\mu,\nu)$, let W_v and W'_v be class 1 Whittaker functions, τ_v be the trivial representation of K_v and $\eta_{1,v}$ (resp. $\eta_{v,2}$) be the characteristic function of the standard lattice in $\mathrm{R}_{E/F}(M_{2n,n}(F_v))$ (resp. $\mathrm{R}_{E/F}(M_{n,1})(F_v)$). Then both W^*_v and W'_v are also class 1 Whittaker functions. On the one hand, by Lemma 2, we have

$$J^*_v(s,W^*_v,W''_v)=\frac{L_v(s+1/2,\sigma^*_v\times\pi_v)}{L_v(2s+1,\pi_v,r_1)}.$$

However, by [Wa2], it follows that

$$V^s_{(\overline{W}'_v,\eta_{1,v})} = L_v(s+1/2,\overline{\pi}_v)\overline{W}'_v.$$

Thus, by Lemma 3, we have

$$J_v(s,W_v,V^s_{(\overline{W}'_v,\eta_{1,v})},R_{\eta_{2,v}}) = \frac{L_v(s+1/2,\overline{\pi}_v)L_v(s+1/2,\sigma_v\times(\overline{\pi}_v\otimes\nu_v))}{L_v(s+1/2,\overline{\pi}_v,r_1)}.$$

Notice that $L_v(s,\overline{\pi}_v,r_1)=L_v(s,\pi_v,r_1)$. Consequently, we obtain

$$\begin{aligned}&\left(\prod_{v\in S} J^*_v(s,W^*_v,W''_v)\right)\frac{L_S(s+1/2,\sigma^*\times\pi)}{L_S(2s+1,\pi,r_1)}\\ &=\left(\prod_{v\in S} J_v(s,W_v,V^s_{(\overline{W}'_v,\eta_{1,v})},R_{\eta_{2,v}})\right)\frac{L_S(s+1/2,\overline{\pi})L_S(s+1/2,\sigma\times(\overline{\pi}\otimes\nu))}{L_S(2s+1,\pi,r_1)}.\end{aligned}$$

By analytic continuation, this equation holds for all $s\in\mathbb{C}$. For each $v\in S$ and a given $s_0\in\mathbb{C}$, the space

$$\{V^{s_0}_{\overline{W}'_v,\eta_{1,v}}|_{M(F_v)}\colon W'_v\in W(\pi_v,\psi_v^{-1}),\ \ \eta_{1,v}\in\mathcal{S}(\mathrm{R}_{E/F}(M_{2n,n})(F_v))\}$$

equals $W(\overline{\pi}_v,\psi_v^{-1})$ (cf. [Wa2]). By Lemmas 5 and 9, we can choose W_v, $V^s_{(\overline{W}'_v,\eta_{1,v})}$ and $\eta_{2,v}$ such that $J_v(s,W_v,V^s_{(\overline{W}'_v,\eta_{1,v})},R_{\eta_{2,v}})=J^*_v(s,W^*_v,W''_v)$ is not identically zero. This completes the proof. ∎

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